Computing Sums of Powers–Christmas Style

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1 Abstract

Inspired by the spirit of Christmas, some random musings happened to lead to a simple method for recursively computing the sum of powers polynomials. We proceed to prove, by direct calculation, that

$$\int x^m dx = \frac{x^{m+1}}{m+1} + C$$

For natural numbers, m.

2 A Partridge in a Pair Tree

We all know the song. It's often used to teach programming because of its extremely repetitive, recursive structure. That's also the reason it becomes annoying to listen to. Nevertheless, let's consider the question: "How many gifts does the lyricist receive in total?"

2.1 The Song

Here are the lyrics, in case you don't know the song:

[Verse 1]
On the first day of Christmas my true love sent to me
A partridge in a pear tree

[Verse 2]
On the second day of Christmas my true love sent to me
Two turtle doves, and
A partridge in a pear tree

⋮

[Verse 12]
On the 12th day of Christmas my true love sent to me
12 drummers drumming
11 pipers piping
10 lords a-leaping
Nine ladies dancing
Eight maids a-milking
Seven swans a-swimming
Six geese a-laying
Five golden rings
Four calling birds
Three french hens

Two turtle doves, and
A partridge in a pear tree

2.2 Taking it Literally

Now, if we're to take these words at face value, we might expect that the lyricist received a total of 12 partridges in 12 (separate?) pair trees. That's because he (or she, etc.) gets a partridge every day. Of course, that probably isn't what's meant, or even the most common interpretation of the lyrics, but let's just roll with it.

The total amount of gifts is represented by the following diagram:

$$\textit{No. of Gifts:} \quad \begin{matrix} & & & & 12 \\ & & & \iddots & \vdots \\ & & 3 & \cdots & 3 \\ & 2 & 2 & \cdots & 2 \\ 1 & 1 & 1 & \cdots & 1 \end{matrix}$$

Where each row represents the type of gift, and each column represents the day. Now, that we've established the problem, it's clear the solution is to sum up the numbers in the diagram.

3 Two Ways to Sum the Diagram

Now, it turns out that there are actually at least two ways to sum up the numbers in the more general diagram:

$$\begin{matrix} & & & & n \\ & & & \iddots & \vdots \\ & & 3 & \cdots & 3 \\ & 2 & 2 & \cdots & 2 \\ 1 & 1 & 1 & \cdots & 1 \end{matrix} \tag{1}$$

One can sum up the columns then add up the column totals, or one can sum up the rows and then add up the row totals. Either way, we expect to arrive at the same total since we've added every number in (1) exactly once. Hence,

$$\sum_{k=1}^{n} (k\text{-th row total}) = \sum_{k=1}^{n} (k\text{-th column total}) \tag{2}$$

Since there are n rows and n columns. Now, let's get expressions for each of those rows, columns:

$$k\text{-th row (from bottom) total} = (n+1-k) \cdot k$$

$$k\text{-th column (from left) total} = \sum_{i=1}^{k} i$$

And so, we may compute the total sum as follows:

$$\text{total (computed by row)} = \sum_{k=1}^{n} [(n+1-k) \cdot k] = (n+1) \sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$$

$$\text{total (computed by column)} = \sum_{k=1}^{n} \sum_{i=1}^{k} i$$

It looks like any way we slice it, we need to know how to sum consecutive integers–in fact, we will see that we will also need to know how to sum consecutive squares, as well. Luckily, this is achievable.

3.1 Sum of Powers Polynomials

Let's examine the sum

$$S = 1 + 2 + 3 + \cdots + n = n + \cdots + 3 + 2 + 1$$

So, adding, we obtain

$$2S = (n+1) + \cdots + (n+1) = n(n+1)$$

Thus,

$$1 + 2 + 3 + \cdot + n = S = \frac{n(n+1)}{2} \tag{3}$$

(3) happens to be what we'll call the first sum of power polynomial. That's because it turns out (indeed, we will see) that there's a polynomial expression for every sum of the first n consecutive integer powers. Now, we'd like to compute $1^2 + 2^2 + 3^2 + \cdots + n^2$ in order to solve out original question. It is at this point that (2) comes back in handy; we have

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \sum_{k=1}^{n} (n+1-k)k$$

We immediately get that

$$\frac{1}{2}(\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k) = (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$$

And a little algebra shows that

$$\sum_{k=1}^{n} k^2 = \frac{2n+1}{3}\sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6} \tag{4}$$

Indeed, (4) is giving us the second sum of powers polynomial. Now we have all the mathematics necessary to attack original question.

3.2 The Return of Christmas

The original question was: "Given the weird, literal interpretation of the lyrics, how many presents did the lyricist receive in total?" The answer is most easily calculated by adding up rows; again,

$$\textbf{Total gifts } = 13\sum_{k=1}^{12} k - \sum_{k=1}^{12} k^2 = 13\frac{12 \cdot 13}{2} - \frac{12 \cdot 13 \cdot 25}{6} = 13(78 - 50) = 4 * 7 * 13 = 364$$

I guess the song makes one more; one for each day of the year–clever lyricist.

4 Beyond Christmas

We did something pretty interesting here; we used (2) as a way to calculate (4) given (3). There's no reason to stop here. The trick is to generalize (1):

$$\begin{array}{ccccc} & & & & n^m \\ & & & \iddots & \vdots \\ & & 3^m & \cdots & 3^m \\ & 2^m & 2^m & \cdots & 2^m \\ 1^m & 1^m & 1^m & \cdots & 1^m \end{array} \tag{5}$$

This diagram allows us to pass from the n-th sum of powers polynomial to the $(n+1)$-th one. Let's perform the calculation.

The first step is to use (2) and (5). Now,

$$k\text{-th row (from bottom) total} = (n+1-k)\cdot k^m$$

$$k\text{-th column (from left) total} = \sum_{i=1}^{k} i^m$$

So, (2) yields

$$\sum_{k=1}^{n}(n+1-k)\cdot k^m = \sum_{k=1}^{n}\sum_{i=1}^{k} i^m \tag{6}$$

Using (6) we now endeavor to prove that the leading coefficient of the n-th sum of powers polynomial is $\frac{1}{n+1}$.

4.1 An Induction Proof

Lemma 1 (Integral-type Sums)**.** *There is is a polynomial, p_m, so that for all natural n,*

$$p_m(n) = \sum_{k=1}^{n} k^m$$

Moreover, p is of degree $(m+1)$ and the leading coefficient is $\frac{1}{m+1}$.

Proof. We proceed by (strong) induction. We have (3) for the case $m=1$, and can immediately check that our claims hold (use $p_0(n)=n$ and everything checks out, as well). Now, for the inductive step, we use (6) and the induction hypothesis; if $p_m(n) = \sum_{k=0}^{m+1} a_k n^k$ and $\hat{p}_{m+1}(n) = \sum_{k=1}^{n} k^{m+1}$,

$$\hat{p}_{m+1}(n) = (n+1)p_m(n) - \sum_{k=1}^{n} p_m(k) = \sum_{j=0}^{m} a_j\left(n^{j+1}+n^j-p_j(n)\right) + a_{m+1}\left(n^{m+2}+n^{m+1}-\hat{p}_{m+1}(n)\right)$$

And so, after a little algebra, we conclude that

$$\hat{p}_{m+1}(n) = \frac{a_{m+1}}{a_{m+1}+1}n^{m+2} + \left[\frac{a_{m+1}}{a_{m+1}+1}n^{m+1} + \sum_{j=0}^{m} a_j\left(n^{j+1}+n^j-p_j(n)\right)\right]$$

And so, indeed, $p_{m+1} = \hat{p}_{m+1}$ is a polynomial of degree $m+2$, as expected. Moreover, since $a_{m+1} = \frac{1}{m+1}$ by hypothesis, then since the stuff inside the brackets is all of degree $m+1$ (also by hypothesis), we can conclude that the leading term of p_{m+1} is

$$\frac{a_{m+1}}{a_{m+1}+1} = \frac{\frac{1}{m+1}}{\frac{1}{m+1}+1} = \frac{1}{m+2}$$

This is what we wanted; the result is true for all natural m by induction.

□

4.2 Some Direct Integrals

Theorem 1 (Integral of natural powers)**.**

$$\int_0^b x^n = \frac{b^{n+1}}{n+1}$$

Proof. Now, let's recall that the Riemann integral of $\int_0^b x^n dx$ is given by the Riemann sum $\lim_{h\to\infty}\sum_{k=1}^{n}\hat{x}_k^k\cdot\frac{b}{h}$. Since x^n is increasing (which can easily be shown by induction), we may conclude

$$\lim_{h\to\infty}\frac{b^{n+1}}{h^{n+1}}\sum_{k=0}^{h-1}k^n \le \lim_{h\to\infty}\sum_{k=1}^{h}\hat{x}_k^k\cdot\frac{b}{h} \le \lim_{h\to\infty}\frac{b^{n+1}}{h^{n+1}}\sum_{k=1}^{h}k^n$$

Now, thanks to (1), we obtain

$$\lim_{h\to\infty} \frac{b^{n+1}}{n+1} + \frac{b^{n+1}}{h^{n+1}} p(h) \leq \lim_{h\to\infty} \sum_{k=1}^{h} \hat{x}_k^k \cdot \frac{b}{h} \leq \lim_{h\to\infty} \frac{b^{n+1}}{n+1} + \frac{b^{n+1}}{h^{n+1}} q(h)$$

where p, q are polynomials of degree n. And so, taking $h \to \infty$, $\frac{1}{h^{n+1}}$ dominates, and those terms vanish. Thus, by the squeeze theorem, we obtain our result, (1).

$$\frac{b^{n+1}}{n+1} = \lim_{h\to\infty} \sum_{k=1}^{h} \hat{x}_k^k \cdot \frac{b}{h} = \int_0^b x^n dx$$

□

The domain of the result can be extended to all reals by leveraging the even/oddness of x^n according to the even/oddness of n and then utilizing the linearity of the integral. We don't bother with that here, though; the hard work is done.